Fundamental Limits of One-Shot Communication

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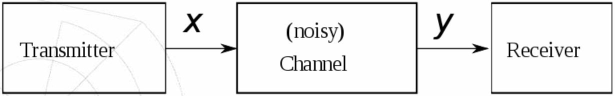
creating and sharing knowledge for telecommunications

Classical Channel Capacity



Shannon's Channel Capacity: $C = \sup_{p_X} I(X, Y)$

- Implicit in the definition:
  - **Arbitrarily large block length** of the channel code
  - **Arbitrarily small error probability** in the decoding process

- Remark: Block Length "=" Number of Channel Uses

Blocklength and Error Probability Analysis

- **Limited number of channel uses**
  - Rate at which the error probability decays to zero: **Error Exponents** (Shannon, Gallager, Berlekamp, 1967)
  - Tight bounds for the **maximal rate for blocklenghts as short as** 100 (Polyanskiy, Poor, Verdú, 2010)
- **Error probability precisely zero**
  - Achievable rates with error probability precisely zero: **Zero-Error Capacity** (Shannon, 1956; Lovász, 1979)

The One-Shot Case

- **One-Shot Capacity** - Maximum number of bits that can be transmitted over the channel if:
  - We can **use the channel only once**
  - We allow the **error probability to go up to a certain user-defined value**

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- Renner, Wolf and Wullschleger (2006) provide **bounds** for the one-shot capacity using Rényi Entropy

Definitions

- A discrete channel is composed of:
  - An input alphabet $\mathcal{X}$ and an output alphabet $\mathcal{Y}$
  - The transition probabilities $\mathcal{P}(Y = y|X = x)$
- A **one-shot communication scheme** over a $P_{Y|X}$ channel is composed of:
  - A **codebook** $\underline{\mathcal{X}} \subseteq \mathcal{X}$
  - A **decoding function** $\gamma : \mathcal{Y} \to \underline{\mathcal{X}}$
- The **maximum error probability** associated with a pair $(\underline{\mathcal{X}}, \gamma)$ is defined as

$$\epsilon_{\underline{\mathcal{X}},\gamma} = \max_{x \in \underline{\mathcal{X}}} \mathcal{P}(\gamma(Y) \neq x | X = x)$$

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Admissibility and Capacity

Definition (Admissible Codebooks)

The pair $(\underline{\mathcal{X}}, \gamma)$ is *maximum-ϵ-admissible* if $\epsilon_{\underline{\mathcal{X}},\gamma} \leq \epsilon$. The set of all ϵ-admissible pairs is denoted by $\mathcal{A}_\epsilon$.

Definition (One-Shot Capacity)

For $\epsilon \in [0,1]$, the *ϵ-maximum one-shot channel capacity* is defined as

$$C_\epsilon = \max_{(\underline{\mathcal{X}},\gamma)\in\mathcal{A}_\epsilon} \log(|\underline{\mathcal{X}}|).$$

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Zero-Error One-Shot Capacity

- The Zero-Error One-Shot Capacity is a particular case of the Zero-Error Capacity (Shannon, 1956; Korner, Orlitsky, 1998):
- The Zero-Error Capacity was fully characterized using a combinatorial approach:
  - Confusion Graph: two input symbols are connected if they can be "confused"
  - Zero-Error Capacity $= \sup_n \alpha(G^n)$
  - One-Shot case: $\alpha(G)$

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- **Is the ϵ-error one-shot capacity significantly different?**

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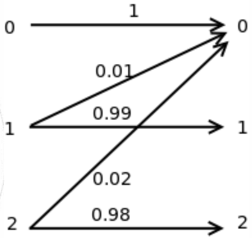
Can we transmit more?



- Zero-Error One-Shot Capacity = 0

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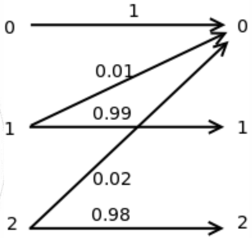

- Zero-Error One-Shot Capacity = 0
- **However**, if we allow for a **small error probability**, we can transmit **more bits** in a single use of the channel:

$$C_\epsilon = \begin{cases} 0 & \text{if } \epsilon < 0.01 \\ 1 & \text{if } 0.01 \le \epsilon < 0.02 \\ \log(3) & \text{if } \epsilon \ge 0.02 \end{cases}$$

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A family of examples

Definition (A Class of Discrete Channels)

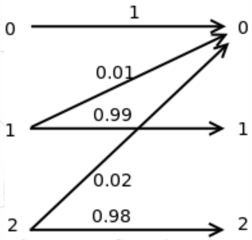
- $\mathcal{X} = \mathcal{Y} = \{0, 1, \dots, n-1\}$
- $\mathcal{P}(Y = 0|X = 0) = 1$ and, with $0 < e_1 < e_2 < \cdots < e_{n-1} \leq 1$, for $i \in \mathcal{X} \setminus \{0\}$,

$$P(Y = y|X = i) = \begin{cases} 1 - e_i & \text{if } y = i \\ e_i & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Lemma

For $e_i \leq \epsilon < e_{i+1}$, *with* $e_0 = 0$ *and* $e_n = 1$, *we have that*

$$C_\epsilon = \log(i+1).$$



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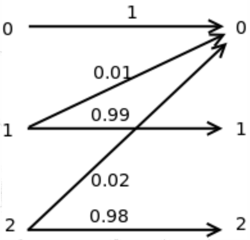
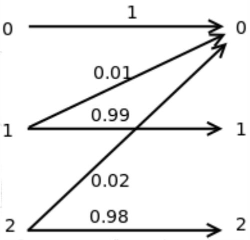


For $\epsilon = 0.01$:

- Input symbol 0: $\gamma^{-1}(0) = \{0\}$, $\gamma^{-1}(0) = \{0, 1\}$, $\gamma^{-1}(0) = \{0, 2\}$



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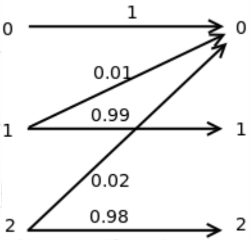

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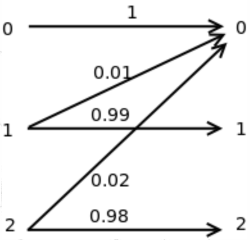

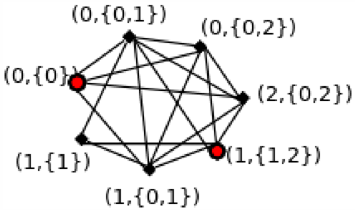

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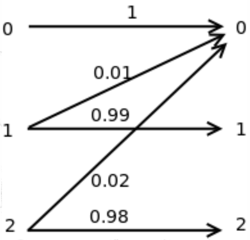

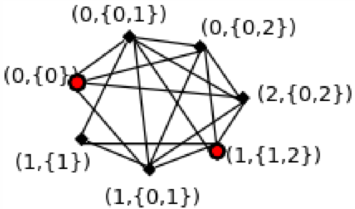

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Look for **Independent Sets**

Maximum-One-Shot Graph

Definition

For each $x \in \mathcal{X}$, let

$$D_\epsilon(x) = \left\{ D \subset \mathcal{Y} : \sum_{y \in D} \mathcal{P}(Y = y | X = x) \geq 1 - \epsilon \right\}$$

Definition (Maximum-One-Shot Graph)

- Nodes: (x, D) with $x \in \mathcal{X}$ and $D \in D_\epsilon(x)$
- (x, D) and (x', D') are connected $\Leftrightarrow x = x'$ or $D \cap D' \neq \emptyset$

Main Result

Theorem

Consider a channel described by $P_{Y|X}$ *and the corresponding one-shot graph* $G_\epsilon = (V, E_\epsilon)$*, with* $\epsilon \in [0, 1)$*. The* ϵ*-maximum one-shot capacity satisfies*

$$C_\epsilon = \log(\alpha(G_\epsilon)).$$

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Main Argument in the Proof:

- **Decoding Function** $\leftrightarrow$ **Independent Set** in the maximum one-shot graph

Complexity

- The one-shot capacity is directly related to an **independent set problem**
- The independent set problem in 3-regular graphs (NP-Hard) can be reduced to an instance of the ϵ-maximum one-shot capacity problem, for $\epsilon < 1/3$

Theorem

The computation of the ϵ-maximum one-shot capacity is NP-Hard, for $\epsilon < 1/3$.

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Conclusions and Future Work

- We provided an **exact characterization of the one-shot capacity** for discrete channels, using combinatorial techniques:
  - **Maximum Error** Probability $\rightarrow$ **Independent Sets**
  - **Average Error** Probability $\rightarrow$ **Sparse Sets**
- We proved that computing the one-shot capacity is **NP-Hard**.

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Conclusions and Future Work

- We presented a family of channels for which the zero-error capacity is null, but by allowing a **small error probability**, we can transmit a **significant larger number of bits**.

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- We presented a family of channels for which the zero-error capacity is null, but by allowing a **small error probability**, we can transmit a **significant larger number of bits**.
- The n-**shot capacity** of a memoryless channel is the **one-shot capacity for the** n**-extension** of the channel
  - First step towards an **exact characterization of capacity in the finite blocklength regime**