Limiting relaxed controls and homogenization of moving interfaces

Gabriele Terrone

Instituto Superior Técnico - Lisboa
Departamento de Matemática - CAMGSD
gterrone@math.ist.utl.pt

UT Austin | Portugal
International Collaboratory for Emerging Technologies, CoLab

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The oscillating front

$$\Gamma^{\epsilon}(t) \subset \mathbb{R}^N$$

configuration at time $t > 0$, of an oscillating front propagating with normal velocity depending on the position.

- Any point of the front is moving in the normal direction;
- The medium has a periodic structure: $\epsilon > 0$ size of the periodic cell;
- Normal velocity: $V(x, x/\epsilon)$ periodic w.r.t. x/ϵ.

Goal:

Describe the asymptotic behavior of $\Gamma^{\epsilon}(t)$ when the structure becomes extremely fine ($\epsilon \to 0$).

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Level set formulation

$\underline{t = 0:}$ $\quad \Gamma^{\epsilon}(0) = \{x \in \mathbb{R}^N \, : \, h(x, x/\epsilon) = 0\}$

$\underline{t > 0:}$ $\quad \Gamma^{\epsilon}(t) = \{x \in \mathbb{R}^N \, : \, u^{\epsilon}(t, x) = 0\}$

u^{ϵ} solves the pde:

$$\begin{cases} \partial_t u^{\epsilon} + V(x, x/\epsilon)|D_x u^{\epsilon}| = 0, & \text{in } (0, +\infty) \times \mathbb{R}^N \\ u^{\epsilon}(0, x) = h(x, x/\epsilon) & \text{for any } x \in \mathbb{R}^N. \end{cases} \qquad (\text{pde}_{\epsilon})$$

Goal: ($\epsilon \to 0$): find $\bar{u}$ such that

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Homogenization of pde's

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There exist $\bar{H}(x, p)$ such that $u^\epsilon(t, x) \to u(x, t)$, unique solution of

$$\begin{cases} \partial_t u + \bar{H}(x, D_x u) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ u(0, x) = \bar{h}(x) & \text{for any } x \in \mathbb{R}^N \end{cases} \qquad (\overline{\text{pde}})$$

$\bar{h}(x) = \min_\xi h(x, \xi)$.

[**Ref:** Bardi - Alvarez 2001 $\to$ 2010]

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Homogenization of pde's

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Associated dynamics

The solution of (pde_ϵ) is [the *value function*]:

$$u^\epsilon(t, x) = \inf\{h(x(t), x(t)/\epsilon)\}$$

among all trajectories $x(t)$ of the [cotrolled] dynamics

$$\begin{cases} \dot{x}(t) = V(x(t), x(t)/\epsilon)a(t) \\ x(0) = x \end{cases} \qquad (\text{dyn}_\epsilon)$$

<u>Controls:</u> for any t, $|a(t)| \leq 1$.

The result

Question: Is there also a controlled dynamics for the effective front? [so we can use its value function to describe the profile of the effective front]

Answer: Yes!

$$\dot{x}(t) \in F(x(t)) \qquad (\overline{\text{dyn}})$$

where...

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$$F(x) = \bigcup_{\mu} \left\{ \int_{\mathbb{R}^N \times \overline{B}(0,1)} V(x,y)a \, \mathrm{d}\mu(y,a) \right\}$$

μ: limiting relaxed controls

- are probability measures encoding the asymtpotic behavior of the oscillating variables:

$$\dot{\xi}(t) = V(x, \xi(t))a(t), \qquad x \text{ frozen;}$$

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Limit dynamics:

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = x \qquad (\overline{\text{dyn}})$$

Theorem

The value function associated to $(\overline{\text{dyn}})$*, that is*

$$\bar{u}(t, x) = \inf\{\bar{h}(x(t)) \;:\; x(\cdot) \text{ solution of } (\overline{\text{dyn}})\}$$

solves $(\overline{\text{pde}})$.

(remember: $\bar{h}(x) = \min_\xi h(x, \xi)$ is the initial data of the limit pde)

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Roadmap

$$\begin{cases} \dot{x}(t) = V(x(t), x(t)/\epsilon)a(t) \\ x(0) = x \end{cases} \xrightarrow[\epsilon \to 0]{?} \dot{x}(t) \in F(x(t))$$

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$$\Big\downarrow \ (u^\epsilon = \text{value function})$$

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$(u^\epsilon$ =value function) $\downarrow$ $\qquad\qquad$ $\downarrow$

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Roadmap

$$\begin{cases} \dot{x}(t) = V(x(t), x(t)/\epsilon)a(t) \\ x(0) = x \end{cases} \longrightarrow \dot{x}(t) \in F(x(t))$$

(u^{ϵ}=value function) $\downarrow$

Theorem: $\bar{u}$= value function $\downarrow$

$$\begin{cases} \partial_t u^{\epsilon} + V(x, x/\epsilon)|D_x u^{\epsilon}| = 0 \\ u^{\epsilon}(0, x) = h(x, x/\epsilon) \end{cases} \xrightarrow[\text{homogenization}]{} \begin{cases} \partial_t u + \bar{H}(x, D_x u) = 0 \\ u(0, x) = \bar{h}(x) \end{cases}$$

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The result

Then, thanks to homogenization [$u^\epsilon \to$solutions of $(\overline{\text{pde}})$] , $\bar{u}$ coincides with the limit of u^ϵ and can be used to represent the limit profile of the front:

$$\Gamma(t) = \{x \in \mathbb{R}^N \,:\, \bar{u}(t,x) = 0\}.$$

Obrigado!